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2-SIDED EMBEDDINGS OF PROJECTIVE PLANES INTO 3-MANIFOLDS

BY
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ABSTRACT. Let M be a nonorientable closed 3-manifold which admits a 2-sided embedding of a projective plane. Then we first prove the following theorem: If M has a Heegaard splitting of genus two, then M is homeomorphic to $P^2 \times S^1$. Next, let M be a nonorientable 3-manifold whose fundamental group is abelian. We verify that if M has a Heegaard splitting of genus two, then M is either the nonorientable 2-sphere bundle over the circle or $P^2 \times S^1$.

1. Introduction. It is well known that there exist infinitely many 3-manifolds which admit 2-sided embeddings of the projective plane P^2. But all such known 3-manifolds other than $P^2 \times S^1$ are not irreducible. Thus we can ask if every irreducible closed 3-manifold, which admits a 2-sided embedding of P^2, is homeomorphic to $P^2 \times S^1$. But this looks hard for the general case and perhaps it is unlikely in general. In this paper, we will establish the affirmative answer to the following special case:

MAIN THEOREM. *Let M be a nonorientable closed connected 3-manifold which admits a 2-sided embedding of the projective plane. If M has a Heegaard splitting of genus two, then M is homeomorphic to $P^2 \times S^1$.*

It will be noticed that by Jaco [**4**] and Lickorish [**5**] the 3-manifold M in the theorem is irreducible and that by Ochiai and Takahashi [**8**] there exist infinitely many nonorientable irreducible closed 3-manifolds with Heegaard splitting of genus two.

By the way, Epstein [**1**] and Hempel [3] proved that if a group G is a finitely generated abelian subgroup of $\pi_1(M)$ for some 3-manifold M, then G is one of: Z, $Z + Z$, $Z + Z + Z$, Z_p, $Z + Z_2$. In particular, if M is nonorientable closed, then $\pi_1(M)$ is either Z or $Z + Z_2$. It is easy to see that $\pi_1(V)$ for the 2-sphere bundle over the circle V (resp. $\pi_1(P^2 \times S^1)$) is Z (resp. $Z + Z_2$). Conversely, we may assert that if M is a 3-manifold of Heegaard genus two and $\pi_1(M)$ is abelian, then M is either V or $P^2 \times S^1$. Indeed, we will verify by the above main theorem and Haken's Theorem [**2, 4**] that if M has a Heegaard splitting of genus two, then M is either V or $P^2 \times S^1$.

We work in piecewise linear category throughout this paper. By $N(Y, X)$, we shall denote a regular neighborhood of subpolyhedron Y in a polyhedron X. S^n, D^n

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denote n-sphere, n-disk, respectively. Closure, interior, boundary are denoted by $\mathrm{cl}(\cdot)$, $\mathrm{Int}(\cdot)$, $\partial(\cdot)$, respectively.

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2. Intersections of a Heegaard surface and a projective plane. Let M be a nonorientable closed 3-manifold. A Heegaard splitting $(M; H_1, H_2)$ for M is a representation of M as $H_1 \cup H_2$, where H_1 and H_2 are homeomorphic nonorientable handlebodies of some fixed genus n and $H_1 \cap H_2 = \partial H_1 = \partial H_2 = F$, a Heegaard surface. A 3-manifold H is a nonorientable (resp. orientable) handlebody of genus n if $H = (Q \times I)\sharp(D_1^2 \times S^1)\sharp \cdots \sharp(D_{n-1}^2 \times S^1)$ (resp. $(D_1^2 \times S^1)\sharp \cdots \sharp(D_n^2 \times S^1)$), where Q, I are a Möbius strip, unit interval, respectively, and $\sharp$ means a disk sum. We remark that $(Q \times I)\sharp(Q \times I) = (Q \times I)\sharp(D^2 \times S^1)$ and every nonorientable closed 3-manifold has a Heegaard splitting and furthermore that by Lickorish [5] there exists the unique 3-manifold V with a Heegaard splitting of genus one.

A properly embedded 2-disk D in a handlebody H of genus n is called a meridian-disk of H if $\mathrm{cl}(H - N(D, H))$ is a handlebody of genus $n-1$. A collection of mutually disjoint n meridian-disks $D_1, \ldots, D_n$ in H is called a complete system of meridian-disks of H if $\mathrm{cl}(H - \bigcup_{i=1}^n N(D_i, H))$ is a 3-disk. Furthermore a collection of mutualy disjoint circles on the boundary of H is called a complete system of meridians of H if it bounds a complete system of meridian-disks of H.

Let F be a Heegaard surface in a 3-manifold M and S be a connected closed surface in M with $F \cap S \neq \varnothing$. We may assume by the general position argument that $F \cap S$ consists of mutually disjoint circles. We remark that all such circles are 2-sided in S even though S is nonorientable. Let us suppose that Δ is a 2-disk in M such that $\Delta \cap S = \alpha$ is an arc in $\partial\Delta$, $\Delta \cap F = \beta$ is an arc in $\partial\Delta$, $\partial\alpha = \partial\beta$ and $\alpha \cup \beta = \partial\Delta$. Then Jaco defined in [4] an isotopy of type A at α performed by sliding α across Δ and past β. See Chapter II in [4] in detail. Such an arc is said to be of type I (resp. type II) if it joins a single circle (resp. different circles) in $F \cap S$.

Let P^2 be a 2-sided projective plane in a 3-manifold M and $(M; H_1, H_2)$ be a Heegaard splitting of genus two for M. Let $F = H_1 \cap H_2$. It will be noticed that $F \cap P^2 \neq \varnothing$, because no handlebodies contain a projective plane.

LEMMA 1. *If $F \cap P^2$ is a single circle, then there exists a projective plane P' in M, with $F \cap P'$ a single circle, and a complete system of meridian-disks of H_1, D_1 and D_2, such that $D_1 \cap Q'$ is an arc and $D_2 \cap Q' = \varnothing$, where Q' is the Möbius strip in P' with $Q' = H_1 \cap P'$.*

PROOF. We may assume without loss of generality that $H_2 \cap P^2$ is a Möbius strip Q. Let D_1' and D_2' be a complete system of meridian-disks of H_2. We remark that $Q \cap (D_1' \cup D_2') \neq \varnothing$; if otherwise, the 3-disk $\mathrm{cl}(H_2 - \bigcup_{i=1}^2 N(D_i', H_2))$ contains the 2-sided Möbius strip Q, but it is impossible. By the general position argument, each component of $Q \cap D_i'$ $(i = 1, 2)$ is either an arc properly embedded in D_i' or a circle. But by the cut-and-paste method [3], all the circles are eliminated. Let α be such an arc in $Q \cap D_i'$. Let us suppose that α bounds a 2-disk in Q. That is α is homotopic in Q to a boundary arc in ∂Q leaving their boundary points fixed. Then by Lemma 5 in [7], there exists a new complete system of meridian-disks of H_2, D_1'' and D_2'' such that the number of components of $Q \cap (D_1'' \cup D_2'')$ is less than the number of components of $Q \cap (D_1' \cup D_2')$. We note that Heegaard theory with respect to the orientable case in [7] also holds aside from slight exceptions in the nonorientable case. Thus we may assume that all such arcs do not bound 2-disks in Q. Furthermore suppose that the arc α is an innermost one in D_i'. Let $C = P^2 \cap F = \partial Q$. Since C bounds the 2-disk $D = H_1 \cap P^2$, C is a circle in F which does not separate F into two components; if otherwise, by Corollary 1.1 in [6] D also separates H_1 into two components H_1^1 and H_2^1, both of which are handlebodies of genus one. Hence H_2 contains a 2-sided closed surface S' with three cross caps. But $\pi_1(H_2)$ is a free group and $\pi_1(S')$ is not and so by the Loop Theorem [3] H_2 contains a 2-sided projective plane. It is impossible. Since C does not separate F, D is a meridian-disk of H_1. By the isotopy of type A at α, a new projective plane P' is obtained. And $P' \cap F$ is a single circle and $P' \cap H_1$ is a Möbius strip Q'. By the way, $D \cap \alpha$ consists of two points $\partial\alpha$, and so two cases happen:

Case 1. α joins two points in ∂D from both sides of D; in this case, we have that $Q' \cap D \neq \varnothing$. And $Q' \cap D$ is an arc α' which does not separate Q' into components. Let $H = \mathrm{cl}(H_1 - N(D, H_1))$ and $D_1 = Q' \cap H$. Then H is a handlebody of genus one and D_1 is a properly embedded 2-disk in H. Hence ∂D_1 is either a meridian of H or a circle homotopic to zero in ∂H. It is easy to see that in each case there exists a meridian-disk D' of H, with $D' \cap Q' = \varnothing$.

Case 2. α joins two points in ∂D from one side of D; in this case, Q' is contained in $H = \mathrm{cl}(H_1 - N(D, H_1))$. Since ∂H is a Klein bottle and $\partial Q'$ is contained in it, by Lickorish [5] there exists a meridian-disk D' of H such that $D' \cap Q'$ is an arc which does not separate Q'.

The proof is complete.

It will be noticed that this lemma holds when the genus of F is greater than 2.

Let D_1 and D_2 be a complete system of meridian-disks of H_1. Suppose that $P^2 \cap F$ is a single circle C, $P^2 \cap H_1 = Q$, $P^2 \cap H_2 = D^2$ and that $D_1 \cap Q$ is an arc which does not separate Q and that $D_2 \cap Q = \varnothing$. Then we have

LEMMA 2. *The manifold M is $P^2 \times S^1$.*

PROOF. Let S be the closed surface obtained by surgery on F along D^2 and let $H = \mathrm{cl}(H_1 - N(D_2, H_1))$. Then H is a nonorientable handlebody of genus one, because Q is a 2-sided Möbius strip properly embedded in H. Hence S is a Klein bottle and by Lickorish [5] S contains only two circles C_1 and C_2, up to isotopy on S, such that they are not homotopic to zero in S. Let $\overline{C}$, $\overline{C}_1$, and $\overline{C}_2$ be the circles illustrated in Figure 1. Then we may assume that $C = \overline{C}$. Furthermore we can assume that $C_1 = \overline{C}_1$ and $C_2 = \overline{C}_2$. But the union $C \cup C_1$ separates F into two components and so the unique Heegaard diagram $(F; \partial D_1 \cup \partial D_2, C \cup C_2)$ is obtained from C. It is easy to see that such a Heegaard diagram gives $P^2 \times S^1$. Hence M is $P^2 \times S^1$.

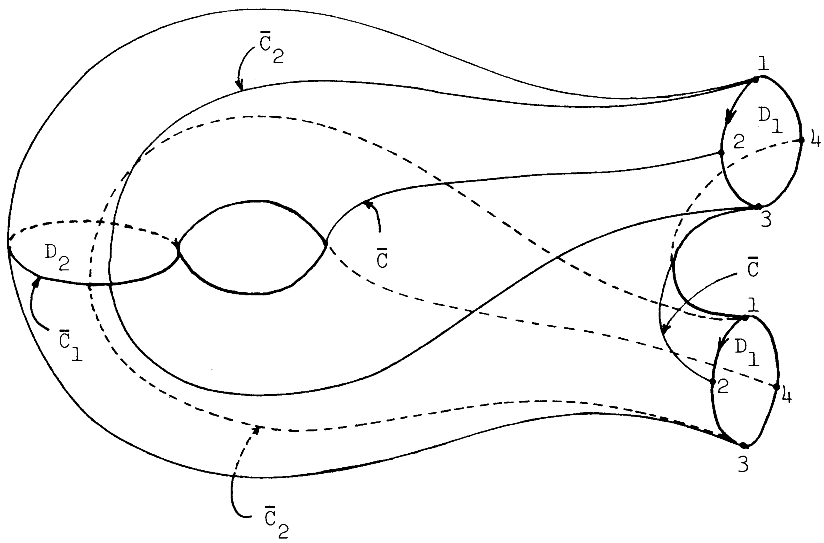

FIGURE 1

By Lemma 1 and Lemma 2, we have

PROPOSITION 1. *If M admits a 2-sided embedding of P^2 such that $P^2 \cap F$ is a single circle, then M is $P^2 \times S^1$.*

Next let us consider the case when $P^2 \cap F$ has many components. By the general position argument, $P^2 \cap F$ consists of mutually disjoint circles, $C_1, C_2, \ldots, C_n$. Then $c(P^2 \cap F) = n$ denotes the number of such circles. Here suppose that each component of $P^2 \cap H_2$ is a 2-disk and that $2 \leq n$. We remark that $P^2 \cap H_1$ is a connected surface with cross cap 1 and with $c(P^2 \cap F)$ boundary circles. Let $P^2 \cap H_2 = D_1 \cup \cdots \cup D_n$ with $\partial D_i = C_i$ $(i = 1, 2, \ldots, n)$ and $S = P^2 \cap H_1$. Let D_1' and D_2' be a complete system of meridian-disks of H_1. We may assume without loss of generality that S is incompressible in H_1 and that $S \cap (D_1' \cup D_2')$ consists of disjoint arcs properly embedded in $D_1' \cup D_2'$. By Lemma 5 in [7], we may assume that every such arc is not homotopic in S to a boundary arc in ∂S leaving boundary points fixed. Such an arc α is said to be of type I.1 (resp. type I.2) if it is of type I and $\mathrm{cl}(S - N(\alpha, S))$ is planar (resp. not planar).

As a trivial observation, we have

LEMMA 3. *If there is an arc α such that α is of type* II *and innermost in $D_1' \cup D_2'$, then M contains a projective plane P', with $c(P' \cup F) < c(P^2 \cup F)$, such that $P' \cap H_2$ consists of disjoint 2-disks.*

We remark that the lemma also holds when $P^2 \cap H_1$ is disconnected.

LEMMA 4. *If α is of type* I.1 *and innermost in $D_1' \cup D_2'$, then M contains a 2-sided projective plane P', with $c(P' \cap F) < c(P^2 \cap F)$, such that $P' \cap H_1$ consists of disjoint 2-disks.*

PROOF. Applying to S the isotopy of type A at α, a new projective plane P_1 is obtained. Since α is of type I.1, we have that $c(P_1 \cap F) = c(P^2 \cap F)$, that $P_1 \cap H_2$ consists of only one Möbius strip and the other 2-disks, and that $S_1 = P_1 \cap H_1$ is a connected planar surface. By a similar method as in the proof of Theorem II.7 in [**4**], there exists a sequence $(S_1, \alpha_1), \ldots, (S_k, \alpha_k)$ for S_1 which gives rise to a sequence of isotopies of S_1 in M, where the first isotopy is of type A at α_1, the second isotopy is of type A at $\alpha_2, \ldots,$ and the kth isotopy is of type A at α_k. Set P' equal to the image P_1 after this sequence of isotopies. Then each component of $P' \cap H_1$ is a 2-disk and $c(P' \cap F) < c(P^2 \cap F)$. This completes the proof.

It will be noticed that Heegaard theory of orientable handlebodies in Chapter II in [**4**] is valid in the case when handlebodies are nonorientable.

LEMMA 5. *If all the circles, $C_1, \ldots, C_n$, are parallel in F, then there exists an arc α such that it is of type* II *and innermost in $D_1' \cup D_2'$.*

PROOF. Let A be an annulus in F such that A contains all the circles. We may assume that $\partial A = C_1 \cup C_n$ and that C_i and C_{i+1} bound a subannulus A_i in A $(i = 1, 2, \ldots, n-1)$, with $A = A_1 \cup \cdots \cup A_{n-1}$ and $\mathrm{Int}(A_i) \cap C_j = \varnothing$ for all j $(j = 1, 2, \ldots, n)$. Now let us suppose that $A \cap (\partial D_1' \cup \partial D_2')$ consists of disjoint arcs in A, each of which joins two points in different components of ∂A; if otherwise, there exists an arc β_1' in $A \cap (\partial D_1' \cup \partial D_2')$, say $A \cap \partial D_1'$, such that it joins two points in one component of ∂A and that it is an innermost one in A. And so there exists an arc β' in $\beta_1' \cap A_k$ for some k such that it joins two points in one component of ∂A_k and that it is innermost in A_k. Let α' be the arc in $\partial A_k = C_k \cup C_{k+1}$, say C_k, which joins two points $\partial\beta'$ with $\mathrm{Int}(\alpha')$ disjoint from $\partial D_1' \cup \partial D_2'$. Let D' be the 2-disk in F, which is bounded by $\alpha' \cup \beta'$. Applying an isotopy to be performed by sliding α' across D' and past β', we get a projective plane P_1 with $P_1 \cap D_2' = P^2 \cap D_2'$. And either $P_1 \cap D_1'$ consists of arcs and only one circle C' or it consists only of arcs. Let $Q = P^2 \cap H_1 = S$ and $Q_1 = P_1 \cap H_1$. In the latter $c(Q_1 \cap D_1')$ is less than $c(Q \cap D_1')$ and in the former it equals that. But in the former the circle C' is eliminated by the cut-and-paste method. Hence we may assume that $c(Q_1 \cap (D_1' \cup D_2'))$ $< c(Q \cap (D_1' \cup D_2'))$. We remark that $c(P_1 \cap F) = c(P^2 \cap F)$ and that all circles in $P_1 \cap F$ are also parallel in F. Repeating the above defined process, at the final step we get a projective plane P_2 such that $c(P_2 \cap F) = c(P^2 \cap F)$, that $c(Q_2 \cap (D_1' \cup D_2'))$ is less than $c(Q \cap (D_1' \cup D_2'))$, and that all circles in $P_2 \cap F$ are parallel in F. Next if some arc in $Q_2 \cap (D_1' \cup D_2')$ bounds a 2-disk in Q_2, then by Lemma 5 in [**7**] we choose a new complete system of meridian-disks of H_2, D_1'' and D_2'', such that every component of $Q_2 \cap (D_1'' \cup D_2'')$ does not bound a 2-disk in Q_2 and that $c(Q_2 \cap (D_1'' \cup D_2'')) < c(Q_1 \cap (D_1' \cup D_2'))$. Consequently we can assume that each arc in $A \cap (D_1' \cup D_2')$ joins two points in both components of ∂A. It will be noticed that A is not disjoint from $D_1' \cup D_2'$, because S is incompressible in H_1. We can assume that $A \cap \partial D_1' \neq \varnothing$ and by Lemma II.9 in [**4**] that there exists at least one arc α of type II in D_1', since the closure of each component of $S -$ $(S \cap (D_1' \cup D_2'))$ is a 2-disk and $c(P^2 \cap F) = n \geqslant 2$. By Lemma 1, we can assume that α is not innermost in D_1'. Let D_{11}' be the closure of one component of $D_1' - \alpha$.

Suppose that D'_{11} contains no arc of type II. Let a be one of the two boundary points of α. Then there exists a segment w in $\partial D'_{11}$ such that $\mathrm{Int}(w)$ is disjoint from C_i for all i ($i = 1, 2, \ldots, n$) and with $w \cap \alpha = a$. Let b be the other point of ∂w and α' be the arc in $D'_{11} \cap S$ with $b \cap \alpha' = b$. We may assume that either a (resp. b) is contained in C_k (resp. C_{k-1}) or in C_1 (resp. C_1 or C_n). Since both points in $\partial\alpha$ lie in different circles but $D'_{11} - \alpha$ contains no arcs of type II, it always holds that for all j ($j = 1, 2, \ldots, n$) there exists an arc α_j in $D'_{11} \cap S$ such that it joins two points in C_j. See Figure 2. Let S_1 be one of the connected components of $\mathrm{cl}(S - N(\alpha_1, S))$. Since S is a Möbius strip with holes, we may assume that S_1 is a connected planar surface with k_1 boundary circles where $k_1 \geqslant 2$. Here all circles in ∂S_1 except one boundary circle can be joined to themselves by disjoint arcs in S_1, which bound no 2-disks in S_1. If $k_1 = 2$, then S_1 is an annulus and does not contain such an arc. Thus we have that $k_1 > 2$. But then there exists a connected planar surface S_2 in S_1 with k_2 boundary circles where $k_1 > k_2 \geqslant 2$. Here S_2 has the similar property to S_1. Repeating this process, at the final step an annulus S_m is obtained. And S_m contains an arc which joins points in one component of ∂S_m but bounds no 2-disks. But it is impossible. As a result, D'_{11} contains at least an arc α'' of type II. We can assume by the above argument that α'' is innermost in D'_1. This completes the proof.

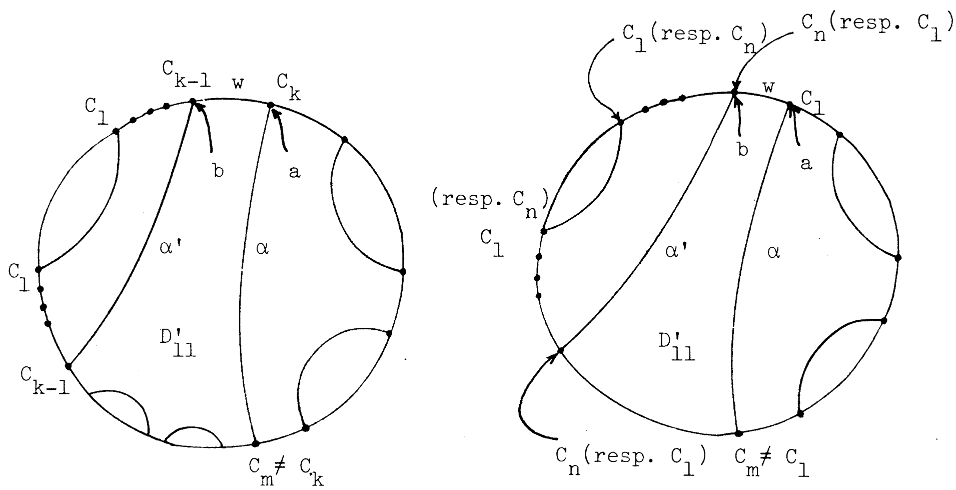

FIGURE 2

Next we will prove the following important fact:

MAIN LEMMA 6. *If all components of* $P^2 \cap H_2$ *are 2-disks, then there exists a 2-sided projective plane* P' *with* $c(P' \cap F) = 1$.

PROOF. We will prove the lemma by induction of $c(P^2 \cap F) = n$. If $n = 1$, then the lemma is valid.

Suppose that $n \geqslant 2$. By Lemma 5, we can assume that both C_1 and C_2 among all the circles $C_1, \dots, C_n$ are not parallel in F and by Lemma 3 and Lemma 4 that all innermost arcs in $S \cap (D_1' \cup D_2')$ are of type I.2. Let α be an innermost arc of type I.2 in $S \cap D_1'$. Then α separates S into two components S_1 and S_2. We remark that one of S_1 and S_2 is planar and the other is a Möbius strip with holes, say S_1. Let $D_1 \cup D_2 \cup \cdots \cup D_n = P^2 \cap H_2$ with $\partial D_i = C_i$ $(i = 1, 2, \dots, n)$ and a, b be the boundary points of α. We can assume that both a and b lie in ∂D_1. Let A be the annulus obtained from D_1 by the isotopy of type A at α and let $\partial A = C' \cup C''$. Then we divide the proof into two parts.

Case (1). C_2 separates F into two components; by Corollary 1.1 in [**6**], D_2 also separates H_2 into two components H_2^1 and H_2^2. Then both H_2^1 and H_2^2 are handlebodies of genus one and D_1 is a meridian-disk of either H_2^1 or H_2^2, say H_2^1, because C_1 is not parallel in F to C_2.

Case (1.a). H_2^1 is orientable; in this case, both C' and C'' bound 2-disks D' and D'' in H_2, respectively, such that they are disjoint from $P^2 \cap H_2$. See Figure 3.1. Since one of C' and C'', say C', bounds S_1, we get a new 2-sided projective plane, $P_1 = S_1 \cup D' \cup D_{11} \cup \cdots \cup D_{1m}$, where D_{1i} is a 2-disk in $P^2 \cap H_2$ and is bounded by some component of $\partial S_1 (i = 1, 2, \dots, m)$. It holds by the construction that $c(P_1 \cap F) < c(P^2 \cap F)$ and then by induction we get a 2-sided projective plane P' such that $P' \cap F$ is a single circle.

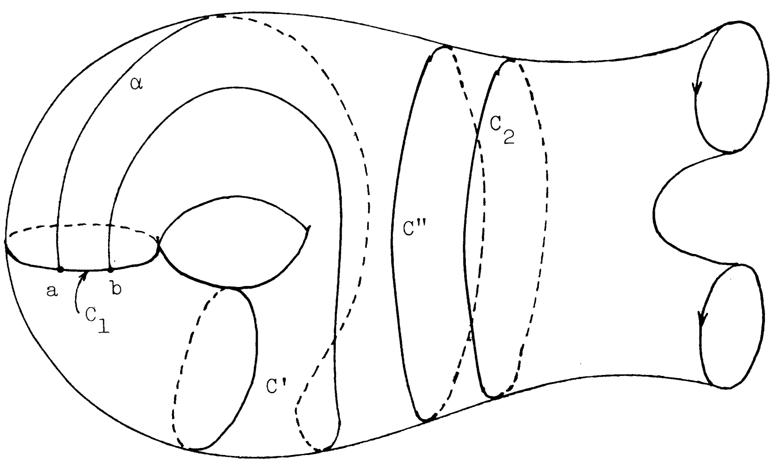

FIGURE 3.1

Case (1.b). H_2^1 is nonorientable; in this case, either both C' and C'' bound 2-disks in H_2 or both are one-sided in ∂H_2^1. See Figure 3.2. The first case is the same as Case (1.a). But the last case does not happen, because both C' and C'' are 2-sided in a 2-sided projective plane in M and so they are homotopic to zero in M.

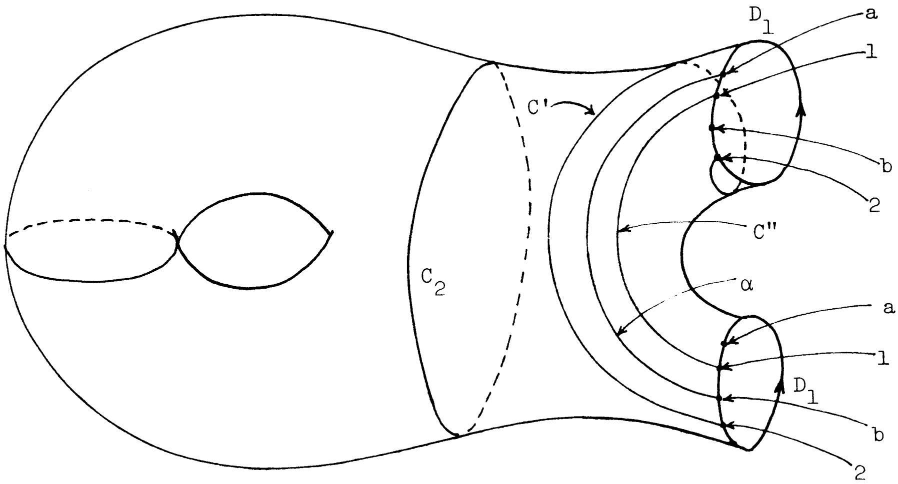

FIGURE 3.2

Case (2). C_2 does not separate F; in this case, two cases happen:

Case (2.a). C_1 does not separate F;

Case (2.a.1). $\mathrm{cl}(H_2 - N(D_2, H_2))$ is orientable; in this case, both C' and C'' also bound 2-disks in H_2. And so this case is the same as Case (1.a).

Case (2.a.2). $\mathrm{cl}(H_2 - N(D_2, H_2))$ is nonorientable; this case is the same as Case (1.b).

Case (2.b). C_1 separates F into two components; in this case, D_1 also separates H_2 into two components H_2^1 and H_2^2. Then both H_2^1 and H_2^2 are handlebodies of genus one and D_2 is a median-disk of either H_2^1 or H_2^2, say H_2^2.

Case (2.b.1). H_2^1 is orientable; in this case, both C' and C'' bound the annulus A and at the same time bound an annulus A' in the boundary of H_2^1, with $A' \cap D_1 = \varnothing$. See Figure 4. To change A to A', we get a new projective plane P_1 with $c(P_1 \cap F) < c(P^2 \cap F)$. Then by induction we get a 2-sided projective plane P' in M such that $P' \cap F$ is a single circle.

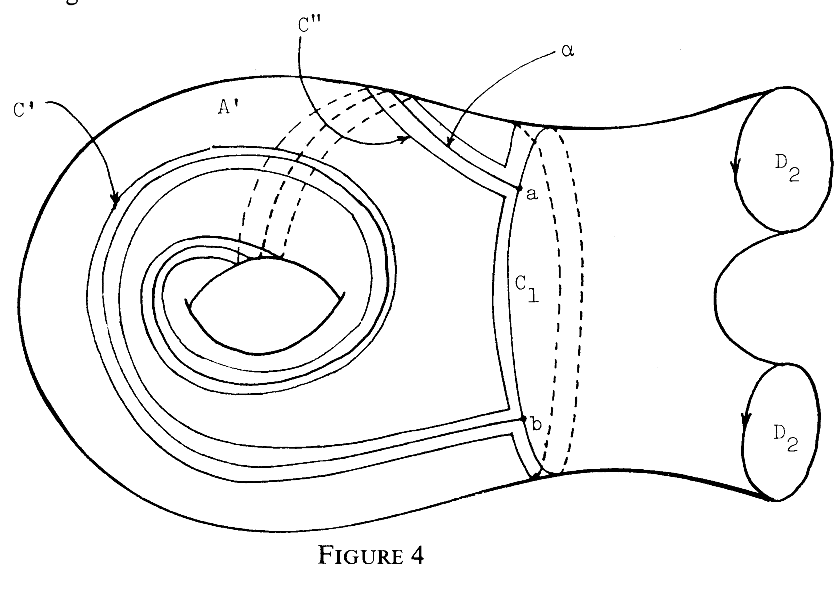

FIGURE 4

Case (2.b.2). H_2^1 is nonorientable; in this case, both C' and C'' bound the annulus A in H_2 and at the same time bound Möbius strips Q_1 and Q_2, respectively, in the boundary of H_2^1. See Figure 5. We remark that $D_1 \cap (Q_1 \cup Q_2) = \varnothing$. It will be noticed that this case depends on the result of Lickorish **[5]**. By the way, one of C' and C'', say C', is one of the boundary circles of S_2 and the other boundary circles of S_2 bound 2-disks $D_{11}, \ldots, D_{1m}$ among 2-disks in $P^2 \cap H_2$. Let $P_1' = S_2 \cup Q_1 \cup D_{11} \cup \cdots \cup D_{1m}$. By deforming the part Q_1 of P_1' into H_1, we get a projective plane P_1 with $c(P_1 \cap F) < c(P^2 \cap F)$. Then by induction we get a 2-sided projective plane P' such that $P' \cap F$ is a single circle.

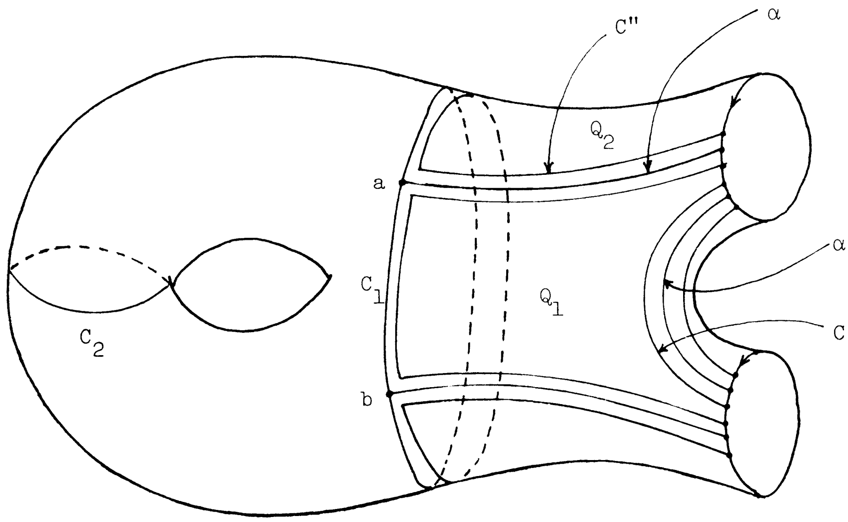

FIGURE 5

The proof is complete.

It will be noticed that it is still possible that C' and C'' may bound 2-disks in Cases (2.b.1) and (2.b.2) but by the same argument as in the Case (1.a) the possibility can be removed from our discussion.

3. Main Theorem and concluding remarks. Let M be a closed connected nonorientable 3-manifold with a Heegaard splitting of genus two, $(M; H_1, H_2)$. Then we have

MAIN THEOREM. *If M admits a 2-sided embedding of a projective plane, then M is homeomorphic to $P^2 \times S^1$.*

PROOF. There exists a 2-sided projective plane P_1^2 in M such that each component of $P_1^2 \cap H_2$ is a 2-disk. Then by Main Lemma 6, we get a 2-sided projective plane P^2 in M such that $P^2 \cap F$ is a single circle. Hence by Proposition 1, M is $P^2 \times S^1$. The proof is complete.

It will be noticed that the generalized form of the Main Theorem does not hold; let M be a nonorientable closed connected 3-manifold with a Heegaard splitting of genus m. In the case when $m = 1$, by Lickorish **[5]** M is the twisted 2-sphere bundle over S^1, V. Since $\pi_1(V)$ is a free group and $\pi_1(P^2)$ is not, by Hempel **[3]** M does not contain a 2-sided projective plane. In the case when $m \geqslant 3$, $V_1 \# \cdots \# V_k \# (P^2 \times S^1) = M$ contains a 2-sided projective plane but it is not $P^2 \times S^1$, where V_i is the twisted 2-sphere bundle over S^1 and $k = m - 2$.

By the way, Tao proved in [9] the following:

Under Poincaré, the following two propositions are equivalent:

(I) Let $\tilde{M}$ be the orientable double covering of a connected closed 3-manifold M. If M is prime, then $\tilde{M}$ is also prime.

(II) $P^2 \times S^1$ is the only connected closed 3-manifold which admits a 2-sided embedding of P^2 and is also irreducible.

In this context, it is an interesting problem to determine all 3-manifolds which are irreducible and admit 2-sided embeddings of P^2. Perhaps such 3-manifolds must have some restricted properties.

Next let M be a nonorientable closed connected 3-manifold such that $\pi_1(M)$ is abelian. Then we have

COROLLARY 1.1. *If M has a Heegaard splitting of genus two, $(M; H_1, H_2)$, then M is either V or $P^2 \times S^1$.*

PROOF. At first, Epstein [1] and Hempel [3] $\pi_1(M)$ is either Z or $Z + Z_2$. Then we have the following two cases:

Case (1). $\pi_1(M) = Z + Z_2$; in this case, by Stallings [10] and Hempel [3], M contains a 2-sided projective plane. Since M has a Heegaard splitting of genus two, by Main Theorem M is $P^2 \times S^1$.

Case (2). $\pi_1(M) = Z$; in this case, by Stallings [10] and Hempel [3] M contains an incompressible 2-sphere S^2. Then by Haken, (see [2]), we may assume that $S^2 \cap F$ is a single circle C. We may assume without loss of generality that C separates F into two components. Hence M has a connected sum decomposition $M_1 \# M_2$ such that both M_1 and M_2 have Heegaard splittings of genus one. Since $\pi_1(M)$ is Z, one of M_1 and M_2, say M_2, is a homotopy 3-sphere. Then M_2 is S^3. By Lickorish [5], M itself is V. The proof is complete.

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